

PIRAMIDA I ZARUBLJENA PIRAMIDA

Slično kao i kod prizme i ovdje ćemo najpre objasniti oznake ...

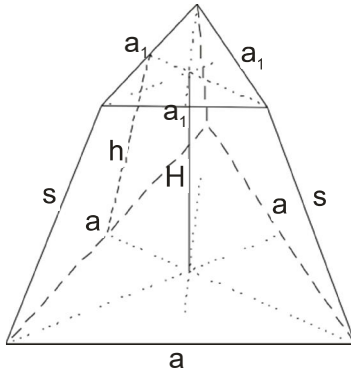
- sa **a** obeležavamo dužinu osnovne ivice
- sa **H** obeležavamo dužinu visine piramide
- sa **h** obeležavamo dužinu visine bočne strane (**apotema**)
- sa **s** obeležavamo dužinu bočne ivice
- sa **B** obeležavamo površinu osnove (baze)
- sa **M** obeležavamo površinu omotača
- omotač se sastoji od **bočnih strana**(najčešće jednakokraki trouglovi) , naravno trostrana piramida u omotaču ima 3 takve strane, četverostrana - 4 itd.
- ako u tekstu zadatka kaže **jednakoivična** piramida, to nam govori da su osnovna ivica i bočna ivica jednake , to jest : **a = s**
- ako u tekstu zadatka ima reč **prava** – to znači da je visina piramide normalna na ravan osnove ili ti , jednostavnije rečeno , piramida nije kriva
- ako u tekstu zadatka ima reč **pravilna** , to nam govori da je u osnovi (bazi) pravilan mnogougao: jednakostraničan trougao, kvadrat, itd.

Dve najvažnije formule za izračunavanje površine i zapremine su:

$$P = B + M \quad \text{za površinu i}$$

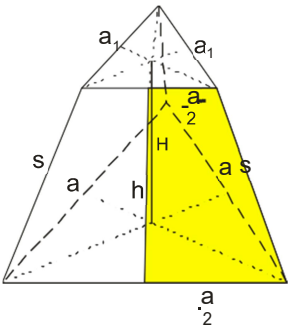
$$V = \frac{1}{3} B \cdot H \quad \text{za zapreminu}$$

PRAVA PRAVILNA TROSTRANA ZARUBLJENA PIRAMIDA

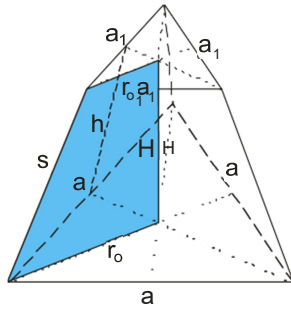


$$P = B + B_1 + M \quad B = \frac{a^2 \sqrt{3}}{4} \quad B_1 = \frac{a_1^2 \sqrt{3}}{4} \quad M = 3 \frac{a + a_1}{2} h$$

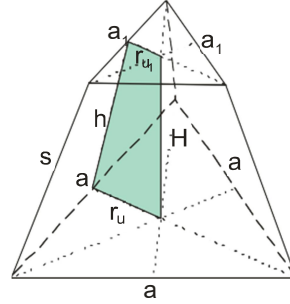
$$V = \frac{H}{3} (B + B_1 + \sqrt{BB_1}) \quad \text{ili} \quad V = \frac{\sqrt{3}H}{12} (a^2 + a_1^2 + aa_1)$$



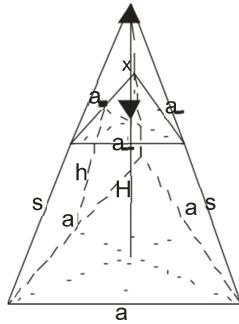
$$\left(\frac{a - a_1}{2} \right)^2 + h^2 = s^2$$



$$\left(\frac{a - a_1}{3} \sqrt{3} \right)^2 + H^2 = s^2$$

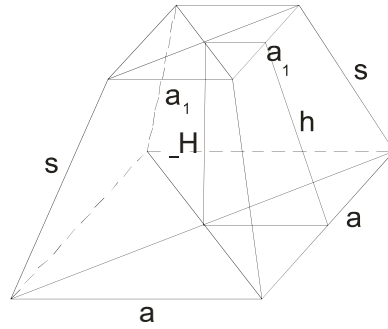


$$\left(\frac{a - a_1}{6} \sqrt{3} \right)^2 + H^2 = h^2$$



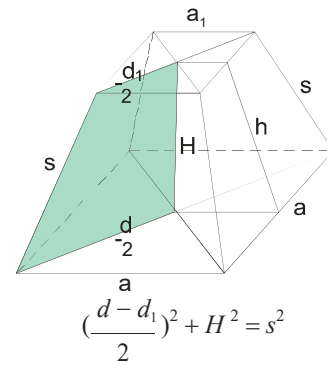
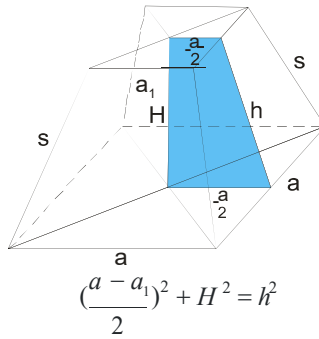
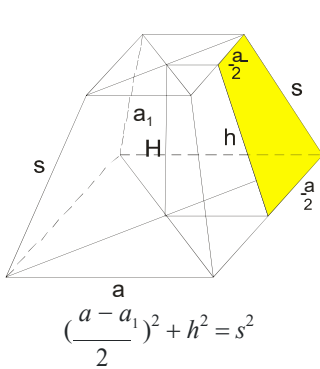
Visina dopunske piramide je: $x = \frac{\sqrt{B_1} H}{\sqrt{B} - \sqrt{B_1}}$

PRAVA PRAVILNA ČETVOROSTRANA ZARUBLJENA PIRAMIDA

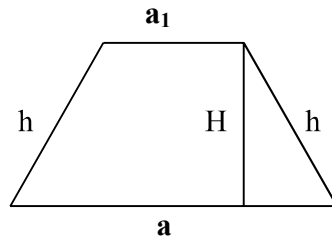


$$P = B + B_1 + M \quad B = a^2 \quad B_1 = a_1^2 \quad M = 4 \frac{a + a_1}{2} h = 2(a + a_1)h$$

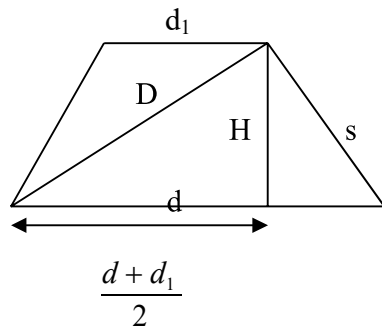
$$V = \frac{H}{3} (B + B_1 + \sqrt{BB_1}) \quad V = \frac{H}{3} (a^2 + a_1^2 + aa_1)$$



osni presek:

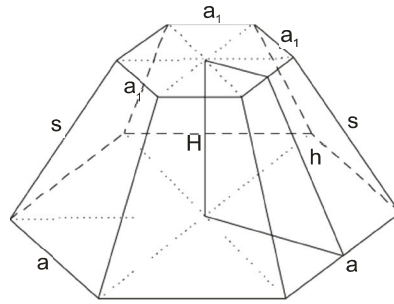


dijagonalni presek:



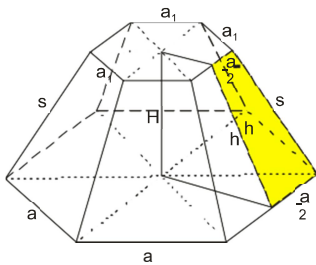
Ako sa x obeležimo visinu dopunske piramide, onda je $x = \frac{\sqrt{B_1}H}{\sqrt{B} - \sqrt{B_1}} = \frac{a_1H}{a - a_1}$

PRAVA PRAVILNA ŠESTOSTRANA ZARUBLJENA PIRAMIDA

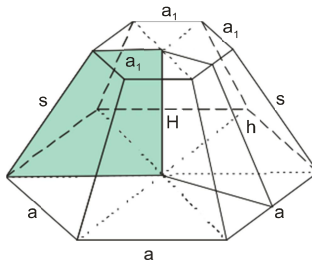


$$P = B + B_1 + M \quad B = \frac{6a^2\sqrt{3}}{4} \quad B_1 = \frac{6a_1^2\sqrt{3}}{4} \quad M = 6 \frac{a+a_1}{2} h = 3(a+a_1)h$$

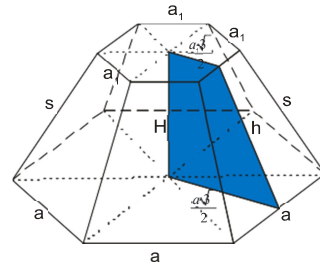
$$V = \frac{H}{3} (B + B_1 + \sqrt{BB_1}) \quad \text{ili} \quad V = \frac{H\sqrt{3}}{2} (a^2 + a_1^2 + aa_1)$$



$$\left(\frac{a-a_1}{2}\right)^2 + h^2 = s^2$$



$$(a - a_1)^2 + H^2 = s^2$$



$$\left(\frac{(a-a_1)\sqrt{3}}{2}\right)^2 + H^2 = s^2$$

Visina dopunske piramide je i ovde: $x = \frac{\sqrt{B_1}H}{\sqrt{B} - \sqrt{B_1}}$

Zadaci

1) Date su osnovna ivica $a = 10\text{cm}$ i visina $H = 12\text{cm}$ pravilne četverostrane piramide. Odrediti njenu površinu i zapreminu.

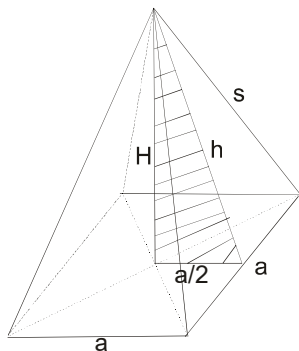
Rešenje:

$$a = 10\text{cm}$$

$$H = 12\text{cm}$$

$$P = ?$$

$$V = ?$$



Prvo ćemo naći visinu h :

$$h^2 = H^2 + \left(\frac{a}{2}\right)^2$$

$$h^2 = 12^2 + 5^2$$

$$h^2 = 169$$

$$\boxed{h = 13\text{cm}}$$

$$P = B + M$$

$$P = a^2 + 2ah$$

$$P = 10^2 + 2 \cdot 10 \cdot 13$$

$$P = 100 + 260$$

$$\boxed{P = 360\text{cm}^2}$$

$$V = \frac{BH}{3}$$

$$V = \frac{a^2 H}{3}$$

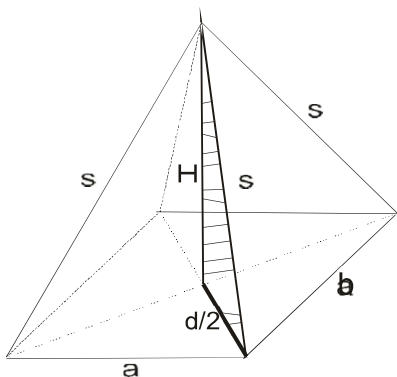
$$V = \frac{10^2 \cdot 12}{3}$$

$$V = 100 \cdot 4$$

$$\boxed{V = 400\text{cm}^3}$$

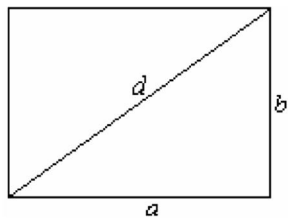
2) Osnova prave piramide je pravougaonik, sa stranicama 12cm i 9cm. Odrediti zapreminu piramide, ako je njena bočna ivica 12,5cm.

Rešenje:



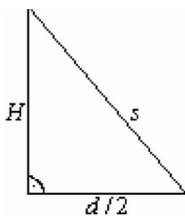
$$\begin{aligned}
 a &= 12\text{cm} \\
 b &= 9\text{cm} \\
 s &= 12,5\text{cm} \\
 \hline
 V &=?
 \end{aligned}$$

Najpre nadjemo dijagonalu osnove (baze)



$$\begin{aligned}
 d^2 &= a^2 + b^2 \\
 d^2 &= 12^2 + 9^2 \\
 d^2 &= 144 + 81 \\
 d^2 &= 225 \\
 d &= 15\text{cm}
 \end{aligned}$$

Sada ćemo naći visinu H iz trougla.



$$\begin{aligned}
 H^2 &= s^2 - \left(\frac{d}{2}\right)^2 & V &= \frac{1}{3}BH \\
 H^2 &= 12,5^2 - 7,5^2 & V &= \frac{1}{3}abH \\
 H^2 &= 100 & V &= \frac{1}{3}12 \cdot 9 \cdot 10 \\
 H &= 10\text{cm} & V &= 360\text{cm}^2
 \end{aligned}$$

3) Osnova prizme je trougao čije su stranice 13cm, 14cm i 15cm. Bočna ivica naspram srednje po veličini osnovne ivice normalna je na ravan osnove i jednaka je 16cm. Izračunati površinu i zapreminu piramide.

Rešenje:

Nadjimo najpre površinu baze preko Heronovog obrasca.

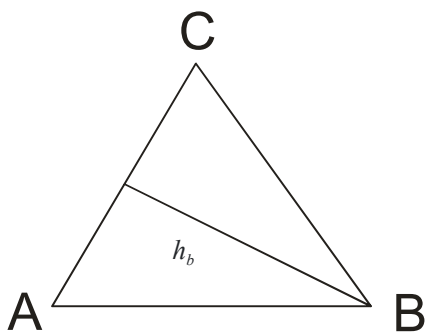
$$a = 13\text{cm}$$

$$b = 14\text{cm} \quad \Rightarrow \quad s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

$$c = 15\text{cm}$$

$$B = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \cdot 7 \cdot 8 \cdot 6} = 84\text{cm}^2$$

Nama treba dužina srednje po veličini visine (h_b) osnove.

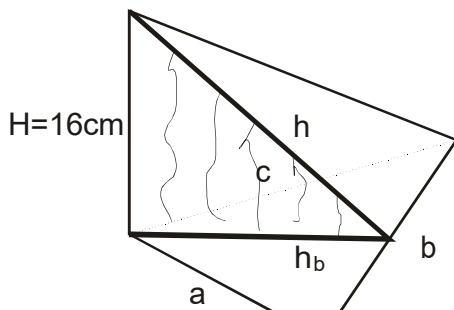


$$P = \frac{b \cdot h_b}{2} \Rightarrow 84 = \frac{14 \cdot h_b}{2}$$

$$84 = 7h_b$$

$$h_b = 12\text{cm}$$

Naći ćemo dalje visinu bočne strane h .



$$h^2 = H^2 + h_b^2$$

$$h^2 = 16^2 + 12^2$$

$$h^2 = 256 + 144$$

$$h^2 = 400$$

$$h = 20\text{cm}$$

Površina piramide je jednaka zbiru površina ova četiri trougla!

$$P = B + \frac{a \cdot H}{2} + \frac{c \cdot H}{2} + \frac{bh}{2}$$

$$P = 84 + \frac{13 \cdot 16}{2} + \frac{15 \cdot 16}{2} + \frac{14 \cdot 20}{2}$$

$$P = 84 + 104 + 120 + 140$$

$$P = 448\text{cm}^2$$

$$V = \frac{1}{3} BH$$

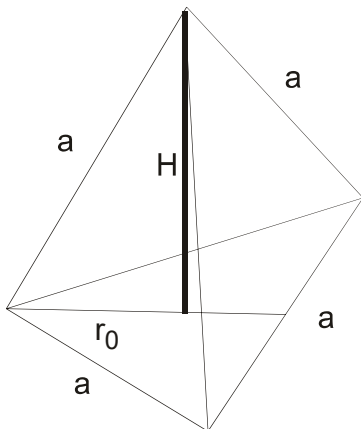
$$V = \frac{1}{3} 84 \cdot 16$$

$$V = 448\text{cm}^3$$

4) Izračunati zapreminu pravilnog tetraedra u funkciji ivice a

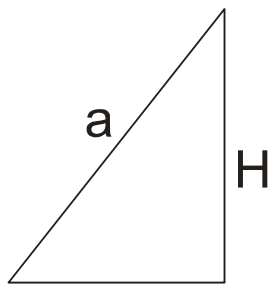
Rešenje:

Tetraedar je pravilna jednakoivična trostrana piramida.



$$V = \frac{1}{3} BH$$

Izvučimo trougao:



$$H^2 = a^2 - \left(\frac{a\sqrt{3}}{3} \right)^2 = a^2 - \frac{a^2 \cdot 3}{9} = \frac{9a^2 - 3a^2}{9} = \frac{6a^2}{9}$$

Dakle:

$$H = \frac{a\sqrt{6}}{3}$$

$$V = \frac{1}{3} \cdot \frac{a^2 \sqrt{3}}{4} \cdot \frac{a\sqrt{6}}{3}$$

$$V = \frac{a^3 \sqrt{18}}{36}$$

$$V = \frac{a^3 \cdot 3\sqrt{2}}{36}$$

PAZI: $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$

$$V = \frac{a^3 \cdot \sqrt{2}}{12}$$

5) Izraziti visinu pravilnog tetraedra u funkciji zapremine V.

Rešenje:

Iskoristićemo rezultat prethodnog zadatka

$$V = \frac{a^3 \sqrt{2}}{12} \quad \text{i} \quad \text{izraziti } a$$

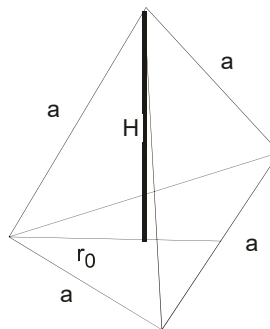
$$a^3 = \frac{12V}{\sqrt{2}}$$

$$a^3 = \frac{12V}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$a^3 = 6\sqrt{2}V$$

$$a = \sqrt[3]{6\sqrt{2}V}$$

$$a = \sqrt[3]{6^6 \sqrt{2^3} \sqrt{V}}$$



Kako je

$$H = \frac{a\sqrt{6}}{3} \text{ to je}$$

$$H = \frac{\sqrt[3]{6^6 \sqrt{2^3} \sqrt{V}} \sqrt{6}}{3}$$

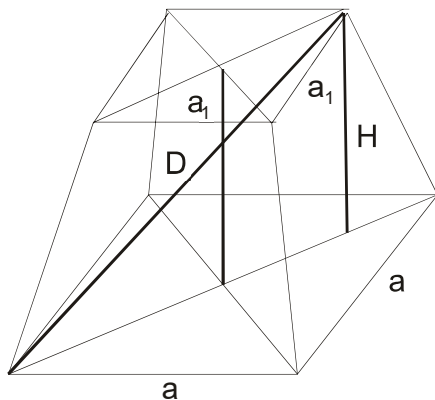
$$H = \frac{\sqrt[6]{6^2} \cdot \sqrt[3]{6^3} \sqrt[6]{2} \sqrt[3]{V}}{3}$$

$$H = \frac{\sqrt[6]{6^5 \cdot 2} \sqrt[3]{V}}{3} = \frac{\sqrt[6]{2^5 \cdot 3^5 \cdot 2} \sqrt[3]{V}}{3}$$

$$H = \frac{2^{\frac{5}{6}} 3^{\frac{5}{6}} \sqrt[3]{V}}{3}$$

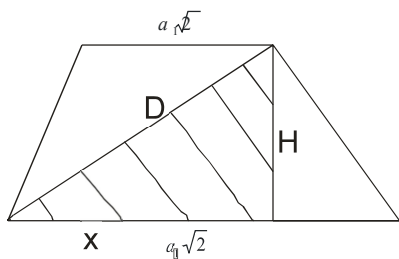
6) Izračunati zapreminu pravilne četverostrane zarubljene piramide ako su osnovne ivice 7m i 5m i dijagonala 9m.

Rešenje:



$$\begin{array}{l} a = 7m \\ a_1 = 5m \\ D = 9m \\ \hline V = ? \end{array}$$

Da bi našli visinu H moramo uočiti dijagonalni presek.



$$x = \frac{a\sqrt{2} + a_1\sqrt{2}}{2}$$

$$x = \frac{7\sqrt{2} + 5\sqrt{2}}{2}$$

$$x = 6\sqrt{2}m$$

$$D^2 = H^2 + x^2$$

$$\begin{aligned} H^2 &= D^2 - x^2 \\ H^2 &= 9^2 - (6\sqrt{2})^2 \end{aligned}$$

$$H^2 = 81 - 72$$

$$H = 9$$

$$H = 3m$$

$$\begin{aligned} V &= \frac{H}{3} (B + B_1 + \sqrt{BB_1}) \\ V &= \frac{3}{3} (a^2 + a_1^2 + aa_1) \end{aligned}$$

$$V = \frac{3}{3} (7^2 + 5^2 + 7 \cdot 5)$$

$$V = 109m^3$$

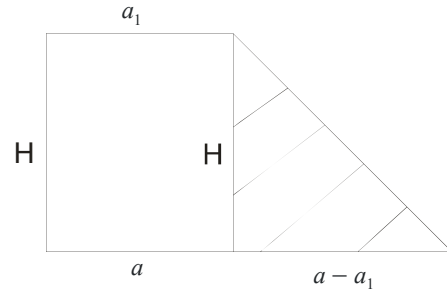
7) Izračunati zapreminu pravilne šestostrane zarubljene piramide ako su osnovne ivice 2m i 1m i bočna ivica 2m

Rešenje:

$$a = 2m$$

$$a_1 = 1m$$

$$s = 2m$$



$$H^2 = s^2 - (a - a_1)^2$$

$$H^2 = 2^2 - 1^2$$

$$H^2 = 3$$

$$H = \sqrt{3}$$

$$V = \frac{H}{3} (B + B_1 + \sqrt{BB_1})$$

$$V = \frac{H}{3} \left(\frac{6a^2 \sqrt{3}}{4} + \frac{6a_1^2 \sqrt{3}}{4} + \frac{6aa \sqrt{3}}{4} \right)$$

$$V = \frac{\sqrt{3}}{3} \cdot \frac{6 \sqrt{3}}{4} (2^2 + 1^2 + 2 \cdot 1)$$

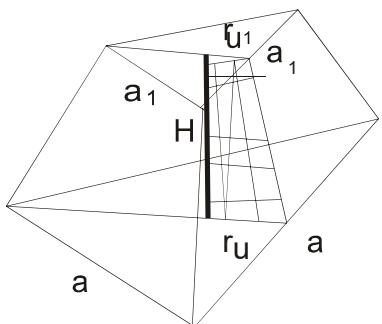
$$V = \frac{3}{2} \cdot 7$$

$$V = \frac{21}{2}$$

$$V = 10,5m^3$$

8) Osnovne ivice pravilne trostrane zarubljene piramide su 2cm i 6cm. Bočna strana nagnuta je prema većoj osnovi pod uglom od 60° . Izračunati zapreminu te piramide.

Rešenje:

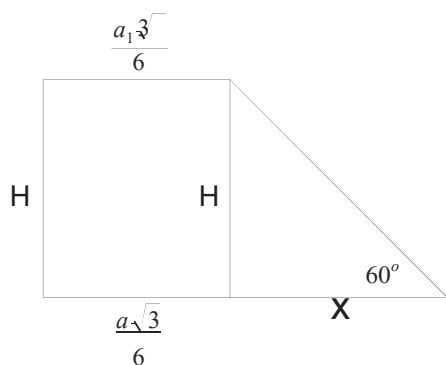


$$a = 6\text{cm}$$

$$a_1 = 2\text{cm}$$

PAZI: Kad se u zadatku kaže bočna strana pod nekim uglom, to je ugao između visina bočne strane i visine osnove!

Izvučimo "na stranu" trapez (pravougli)



$$x = \frac{a\sqrt{3}}{6} - \frac{a_1\sqrt{3}}{6} = \frac{6\sqrt{3}}{6} - \frac{2\sqrt{3}}{6} = \frac{4\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$$

$$\text{tg}60^\circ = \frac{H}{x} \Rightarrow H = x \cdot \text{tg}60^\circ = \frac{2\sqrt{3}}{3} \cdot \sqrt{3} = 2\text{cm}$$

$$V = \frac{2\sqrt{3}}{3} (6^2 + 2^2 + 6 \cdot 2)$$

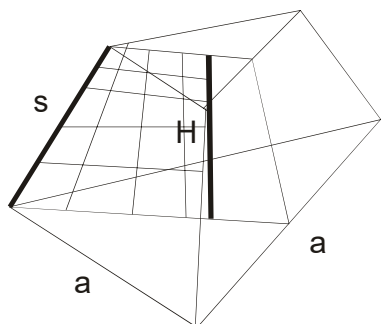
$$V = \frac{\sqrt{3}}{6} (36 + 4 + 12)$$

$$V = \frac{\sqrt{3}}{6} \cdot 52$$

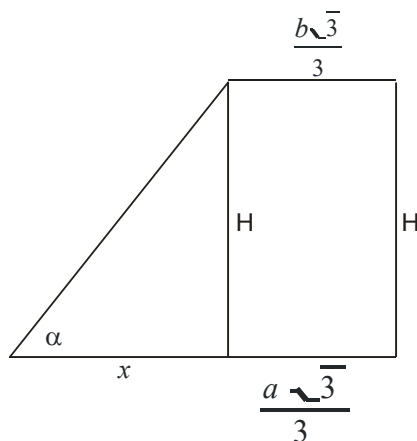
$$V = \frac{26\sqrt{3}}{3} \text{ m}^3$$

9) Bočne ivice pravilne trostrane zarubljene piramide nagnute su prema ravni osnove pod uglom α . Osnovne ivice piramide su a i b ($a > b$). Odrediti zapreminu piramide.

Rešenje:



Izvučimo obeleženi trapez, iz njega ćemo naći visinu!



$$x = \frac{a\sqrt{3}}{3} - \frac{b\sqrt{3}}{3} = \frac{(a-b)\sqrt{3}}{3}$$

$$\operatorname{tg} \alpha = \frac{H}{x}$$

↓

$$H = x \operatorname{tg} \alpha = \frac{(a-b)\sqrt{3}}{3} \cdot \operatorname{tg} \alpha$$

$$V = \frac{H}{3} \left(\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} + \frac{ab\sqrt{3}}{4} \right)$$

$$V = \frac{1}{3} \frac{(a-b)\sqrt{3}}{3} \cdot \operatorname{tg} \alpha \cdot \frac{\sqrt{3}}{4} (a^2 + b^2 + ab)$$

$$V = \frac{(a-b)\operatorname{tg} \alpha}{12} (a^2 + b^2 + ab)$$

Kako je $(a-b)(a^2 + b^2 + ab) = a^3 - b^3$

$$V = \frac{(a^3 - b^3)\operatorname{tg} \alpha}{12}$$

